Spontaneous Symmetry Breaking

Kohtaro Yamakawa*

December 24, 2020

1 Introduction

Emergent phenomena describes a macroscopic behavior of a system that does not appear at the microscopic level. The itinerant nature of electrons in metals, superconductivity, Bose-Einstein condensation, and all modern topics of condensed matter are examples of emergent phenomena that are not properties of a single electron. A notable example is that of Spontaneous Symmetry Breaking (SSB), where a stable state of a system seems to break the symmetries of its own Hamiltonian.

Consider Euler's strut in Figure 1, where a heavy symmetric weight is placed on a solid uniform rod and the rod breaks.

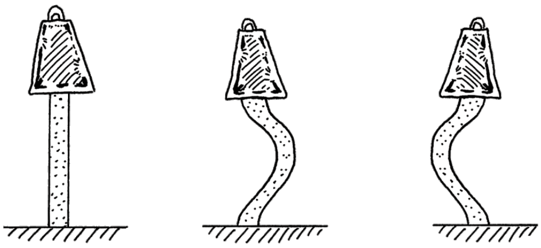

Figure 1: The Euler strut[1]

Before the weight is placed on the rod, the system is mirror symmetric over the vertical axis. However, after the weight is placed, the rod must break in one direction or the other, breaking the mirror symmetry. This is an example of a phase transition in which symmetry is broken, spontaneously. While doing this experiment results in a broken rod bent, there is an equal chance of the rod breaking in either direction and, in either case, the energy of the broken rod would be the same. For the system to preserve symmetry overall, both possibilities must exist and there must be a symmetry among the possible end states.

The SSB Principle is far more than just a new perspective in understanding phase transitions. As we shall soon see, the resulting stable asymmetric states must satisfy certain properties and generate a new mode that has physical, observable consequences. It is a powerful tool that treats problems in statistical mechanics and particle physics on the same footing and creates a unique correspondence between the two fields.

1.1 History

The idea of spontaneous symmetry breaking is not, however, old. While he had not fully understood its significance, Heisenberg had used SSB in his nonlinear unified field theory of magnetism[2]. Even before that, Curie and Weiss used symmetry to describe the phase transition in ferromagnets [3].

*yamakawa.kohtaro@columbia.edu

It was not until the theoretical work of Yoichiro Nambu[4] that the phenomena was understood in its full generality. He first formulated the SSB principle in explaining the preservation of gauge invariance in BCS Theory of Superconductivity[5]. The theory implied a Cooper pair condensate ground state, where electrons are paired together, with a fixed phase which seemed to violate the local gauge symmetry of electromagnetism. Nambu showed that the breaking of the symmetry led to his namesake Nambu-Goldstone massless modes which, when coupled to the gauge field, created a massive gauge field. This not only allowed him to prove that the gauge symmetry was never broken, but also show that BCS theory was in fact equivalent to the Ginzburg Landau theory of phase transitions.

Nambu had the insight to use SSB to understand the chiral symmetry breaking of massless fermion fields, and used the Nambu Goldstone formalism to discover the massless limit of pions. Suddenly, SSB was no longer a concept but a tool to predict observable masses in high energy physics.

2 Spontaneous Symmetry Breaking

Nambu described SSB in an infinite system[1] completely by the following properties in a field theoretic formulation [6].

(i) **Symmetry Broken:** The ground state Ω is degenerate and transforms as a representation of the symmetry group G.

(ii) **Existence of massless modes:** There exist one massless Nambu Goldstone (NG) boson for every globally broken symmetry. These massless modes may be thought of as transformations between asymmetric ground states.

(iii) **Superselection Rules:** No local observables can connect the Hilbert spaces built between different ground states.

The fact that symmetry is preserved regardless of the selection may be better seen by stating properties (*i*) and (*ii*) algebraically. The fact that the ground state transform as a representation of G means that for the set of ground states Ω, each labeled by $\{i\}$, the action of G on a ground state i is another ground state i'.

$$\forall g \in G, \ g \circ i = i'$$

In other words, Ω is a group orbit of G. When the system is in a ground state $i \in \Omega$, the symmetry of the Hilbert space reduces from G to a subgroup G_Ω. The generators Q of the coset $S = \exp[Q] \in G/G_\Omega$ constitute the Nambu Goldstone modes. This Noether charge Q of our "broken" symmetry is observable in the original Hilbert space H_Ω, restoring the overall symmetry.

The last property implies that any and all states must be constructed from operators on the asymmetric vacua/ground state so one cannot encounter observables from other asymmetric ground states. To gain better intuition as to why this is true, we provide the proof to (*iii*) [7].

2.1 Proof of Super Selection Rules

Let us for simplicity consider a theory with $\mathbb{Z}_2$ symmetry, $\phi \to -\phi$ where we denote $\Omega_\pm$ as the two symmetric ground states. Then, the vacuum matrix elements of the Hamiltonian are of the form,

$$\langle \Omega | H \Omega \rangle = \begin{pmatrix} \langle \Omega_+ | H \Omega_+ \rangle & \langle \Omega_+ | H \Omega_- \rangle \\ \langle \Omega_- | H \Omega_+ \rangle & \langle \Omega_- | H \Omega_- \rangle \end{pmatrix}$$

[1] You may see in our proof of the super selection rules that such a condition is necessary.

By symmetry, the diagonal terms (a) must be equal, as are the off-diagonal terms (b).The eigenvalues of the Hamiltonian are $a \pm |b|$ corresponding to eigenstates $\Omega_+ \pm \Omega_-$. b can be understood to be probability amplitude of tunneling between ground states $b \sim \exp[-CV]$ where C is a constant and V is the volume of the system. Therefore, b is miniscule for macroscopic systems.

If we consider an external symmetry breaking perturbation H', its diagonal elements should differ far more than b, b' meaning that the vacuum eigenstate of the perturbed Hamiltonian is much closer to Ω_+ or Ω_- than any linear combination. Therefore, the vacuum eigenstate stable under asymmetric perturbations are themselves asymmetric Ω_+ OR Ω_-.

In the limit of an infinite system, all off-diagonal terms become 0. In this limit, the vacua may be considered as the zero momentum eigenstates. Then, for local operators A, B and position x,

$$\begin{aligned}\langle u|A(x)B(0)v\rangle &= \sum_w \langle u|A(0)w\rangle\langle w|B(0)v\rangle \\ &\quad + \int \mathrm{d}^3p \sum_N \langle u|A(0)|N,\vec{p}\rangle\langle N,p|B(0)|v\rangle e^{-ipx}\end{aligned}$$

where we separated the sum over orthonormalized states between continuum and discrete states. The latter integral term dies off as $|x| \to \infty$ due to the Riemann-Lebesgue theorem so that,

$$\langle u|A(x)B(0)|v\rangle \mapsto \sum_w \langle u|A(0)w\rangle\langle w|B(0)|v\rangle$$

The same argument holds true if we were to switch A, B and since causality implies that for $x \neq 0$, $[A(x), B(0)] = 0$ and the observables $\langle u|A(0)|v\rangle$ and $\langle u|B(0)|v\rangle$ must commute. This means that in infinite volume, any Hamiltonian made of local operators A, B of a certain vacua has vanishing cross term matrix elements with any other vacua. It follows that when we are in one asymmetric ground state, the other ground states cannot appear in any local observable.

2.2 Differences in Formulation

In the above descriptions, we have freely used the terms vacua and ground state equivalently due to the similar treatment of SSB in particle physics and statistical mechanics. Much can be said about the connection between the two fields especially with the similarities in the path integral with the summation over all configurations in the partition function. There are, however, significant differences that must be addressed.

In equilibrium statistical mechanics, SSB is usually discussed in the context of symmetry breaking phase transitions. Perhaps the most well known method of analyzing such processes was invented by Landau[8], who postulated that the free energy of a system must always be continuous and obey the symmetry of the Hamiltonian. One characterizes a continuous phase transition with a phenomenological local variable, called the order parameter, that is nonzero in the ordered or "symmetry broken" phase and zero in the disordered phase. During the symmetry breaking process, the order parameter non-analytically becomes non-zero and one may perturbatively analyze this critical point by expanding the free energy around this point, taking care to respect the symmetries of the full system. The local property of the order parameter is necessary to describe the asymmetric ground states stable in the thermodynamic limit, as seen in the derviation of the superselection rules.

For example, in the case of an Ising model transitioning from a paramagnet to a ferromagnet, the magnetization m becomes nonzero at some temperature T_c. Therefore, since the model has $\mathbb{Z}_2$ symmetry, the free energy has the form,

$$\psi(m) = tm^2 + um^4 + vm^6 + \dots$$

Since the equilibrium ground state is characterized by the absolute minimum of the free energy, the process of symmetry breaking is a shift in the ground state from $m = 0$ to $m \neq 0$. By changing the tuning parameter, the temperature t in this case, the form of the free energy evolves to make nonzero m states energetically favorable to $m = 0$ state. However, since the free energy must respect the symmetry of the system, the m states must also. In the Ising model case, this means a $-m$ state exists for every nonzero m ground state.

The above mean field theory approach allowed us to describe the first property of SSB. In order to see Nambu Goldstone modes, one must introduce fluctuations in the order parameter $m \to m + \delta m$, as in Ginzburg Landau Theory. We are then able to derive massless (gapless) excitations which in the limit $k \to 0$ transform the ground state into other degenerate states. The superselection rules (*iii*) come out in statistical mechanics in a very similar manner as the proof in the previous section.

However, Landau's symmetry breaking formalism does not describe all phase transitions. There exist phase transitions described by *nonlocal* topological order parameters whose phases are dictated by the asymptotic behavior of correlation functions, as in the Berezinski-Kosterlitz-Thouless transition [9] in which the binding of topological defects cause the spin spin correlations function of the XY model to decay as a power law instead of exponentially. This has been found to occur in thin film superconductors and superfluid helium. These phase transitions may still exhibit spontaneous symmetry breaking. Such examples extend to particle physics as well [10] .

In describing the above, we may easily point out a few similarities between particle physics and statistical mechanics. The asymmetric vacuum state stable against external field perturbations translates to the equilibrium ground state robust against thermal (or quantum) fluctuations. The Nambu Goldstone bosons become gapless excitations. Two fields of understanding systems with infinite degrees of freedom seem to have arrived to the same conclusion on stability in symmetry breaking.

2.3 Nambu Goldstone Modes

Perhaps the best known examples of SSB are its massless modes. However, if superselection rules imply that there exist no observables from other ground states, then if we are living in one asymemtric ground state, we cannot actually "use" these massless modes to transform to other ground states. NG modes nonetheless are physical. We first present the theorem behind the existence of the NG modes, Goldstone's theorem[11].

Theorem 2.1. *The breaking of a global,continuous symmetry in the absence of long ranged order/interactions results in a mode in the spectrum whose energy vanishes as its wave number approaches zero.*

While we shall not present its derivation[7], the proof is constructive and one can always easily find at least one such massless excitation in a system exhibiting SSB. An examples of such NG modes are magnons in ferromagnets, the quantization of spin waves. These NG modes create fluctuations in the ground state which, for systems with dimension two, destroy long range order according to the Mermin-Wagner-Hohenberg-Coleman theorem. Its proof was produced separately by Coleman in quantum field theory [12] and by Mermin and Wagner in statistical physics [13].

When the conditions of the above theorem are not met, the result is, obviously, not guaranteed. If a field with energy μ were to break the symmetry, we would have a mode with gap μ. In a different flavour, when the symmetry is broken explicitly due to a weak coupling to gauge degrees of freedom, then we have a pseudo NG boson with a small mass gap. This is what happens in BCS Theory.

In a superconductor, the ground state has a definite phase and seems to break global $U(1)$ symmetry. The subsequent gapless NG mode in the superconductor couples to the photon of electromagnetism (an internal field/gauge degree of freedom), and creates a massive photon. This can be seen in the Ginzburg-Landau Theory formulation,

$$
\begin{aligned}
F &= \frac{1}{2\mu_0}\left(\nabla \times A\right)^2 + \frac{\hbar^2}{2m^*}|\psi|^2\left(\nabla\phi - \frac{e^*}{\hbar}A\right)^2 + \frac{\hbar^2}{2m^*}\left(\nabla|\psi|\right)^2 + \frac{1}{2}r|\psi|^2 + \frac{1}{4}u|\psi|^4 \\
&= \frac{1}{2\mu_0}\left(\nabla \times \tilde{A}\right)^2 + \frac{\left(e^*\right)^2}{2m^*}|\psi|^2\left(\tilde{A}\right)^2 + \frac{\hbar^2}{2m^*}\left(\nabla|\psi|\right)^2 + \frac{1}{2}r|\psi|^2 + \frac{1}{4}u|\psi|^4
\end{aligned}
$$

where $\tilde{A} \equiv A - \frac{\hbar}{e^*}\nabla\phi$. This new vector field is gauge invariant and has "eaten up" the ϕ field, giving it mass. This phenomenon has come to be known the Anderson-Higgs mechanism and is a general mechanism in which gauge fields "acquire" mass.

3 Conclusion

SSB helps us understand statistical mechanics and particle physics in a general framework. Symmetry no longer is merely a property of a system, but proves essential in describing its dynamics. While the critical behavior near symmetry breaking processes constituted a difficulty in perturbative methods, SSB allows us to carefully analyze the physics despite its non-analyticity.

The author would like to thank his collaborator John Staunton for useful discussions and to Professor Uemura for his help in the preparation of this paper.

References

[1] T. Lancaster and S. Blundell. *Quantum Field Theory for the Gifted Amateur.* Oxford Scholarship Online, 2014.

[2] H. Mitter S. Schlieder H-P. Duerr, W. Heisenberg. and K. Yamazaki. *Z. f. Naturforschung 14A*, 59:441, 1954.

[3] P.Curie and P. Weiss. *J. Phys. Theory. Appl.*, 6, 1907.

[4] Y. Nambu. *Rev. Modern Physics*, 81:1015, 2009.

[5] Y. Nambu. *Phys. Rev.*, 117:648, 1960.

[6] Y. Nambu. *Nuclear Physics*, A638:35c–44c, 1998.

[7] S. Weinberg. *The Quantum Theory of Fields: Volume 2.* 1996.

[8] L. Landau. *Zh. Eksp. Teor. Fiz*, 7, 1937.

[9] J. Kosterlitz and D. Thouless. *J. Phys. C*, 5(11), 1973.

[10] J. Greensite and K. Matsuyama. *Phys. Rev. D.*, 98, 2018.

[11] A. Beekman, L. Rademaker, and J. van Wezel. *SciPost Phys. Lect. Notes*, page 11, 2019.

[12] S. Coleman. *Commun.Math. Phys.*, 1973.

[13] N. Mermin and H. Wagner. *Phys. Rev. Lett.*, 17, 1966.