Conformal Symmetry and 2D Renormalization Group*

Kohtaro Yamakawa
University of California Berkeley
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A review of conformal symmetry, conformal invariance, and conformal field theory in the context of two dimensional renormalization group flow and the classification of two dimensional universality classes.

INTRODUCTION

Conformal Symmetry is a type of symmetry under transformations which encompass rotations, translations, and (local) dilations. It has found remarkable application in the study of criticality of two dimensional systems. The goal of this paper is to define conformal transformations and display its power in two dimensional systems. Through this, we will explore the connection between scale and conformal invariance, the central charge/conformal anomaly, and the connection between renormalization group (RG) flow with conformal symmetry.

I. CONFORMAL TRANSFORMATIONS

Given the metric tensor $g_{\mu\nu}$ in a space-time of dimension d, a conformal transformation (CT) is a coordinate transformation $x \to x'$ that leaves the metric invariant up to a number.

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x) \tag{1}$$

A CT is a transformation that locally corresponds to a translation, rotation, and/or dilation transformation. Applying this constraint on a general infinitesimal transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$ implies that it is only conformal if, $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu}$ for some function $f(x)$. Assuming a flat (Euclidean) metric, this statement can be simplified to,

$$(d-1)\partial^2 f = 0 \tag{2}$$

It is here we see that dimensions play a large role in conformal field theories. While any function f is conformal in one dimension, the case for $d = 2$ and $d \geq 3$ are very different. In the latter $d \geq 3$ case, $\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho$ where one may derive that the conformal transformations consist of translations, dilations, rigid rotations, and special conformal transformations (SCT):

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2} \tag{3}$$

While seemingly complicated, the SCT may be rewritten as $x'^\mu / x'^2 = x^\mu / x^2 - b^\mu$, a translation followed by an inversion.

The generators of the global conformal transformations form the conformal algebra and may be rewritten to observe the SO(d+1,1) Lorentz Lie Algebra commutation relations [1]:

$$J_{ab}, J_{cd} = i\left(\eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}\right) \tag{4}$$

Since SO(d+1,1) has $\frac{1}{2}(d+2)(d+1)$ parameters, it takes just as many parameters to fully specify a general conformal transformation for $d \geq 3$.

However, conformal transformations in $d = 2$ case are more complicated. Since all analytic mappings of the complex plane onto itself are conformal, the set of all CT is infinite dimensional. While it takes infinitely many parameters to specify the two dimensional conformal transformation, not all such analytic mappings can form a group. Hence, we define the special conformal group as the set of global CT: transformations that are invertible and map the Riemann sphere onto itself. They are isomorphic to SL(2, $\mathbb{C}$) and are the set of projective transformations,

$$f(z) = \frac{az + b}{cz + d} \tag{5}$$

with $ad - bc = 1$.

Local conformal transformations are then the set of non-invertible or non-holomorphic mappings. Formally, transformations in the local conformal group are generated by the conformal or Witt algebra.

II. CONFORMAL FIELDS

Extending the notion of conformal transformations to classical fields in two dimensions, we denote a quasi-primary field as a field that transforms under a global conformal transformation of the form $z \to w(z)$, $\bar{z} \to \bar{w}(\bar{z})$ as

$$\phi'(w, \bar{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}) \tag{6}$$

for conformal dimensions $(h, \bar{h})$. Here we have used holomorphic/anti-holmorphic coordinates with $z = z_1 + iz_2$ and $\bar{z} = z_1 - iz_2$ with $z_1, z_2 \in \mathbb{R}$. If it transforms in this

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way under local conformal transformations as well, it is called a primary field.

For $d > 2$, a spinless field $\phi(x)$ transforms under a global CT as [2],

$$\phi(x) = \Lambda(x)^{\Delta/2}\phi(x) \tag{7}$$

where Δ is the scaling dimension of the field and $\Lambda(x)$ is the scale factor. These fields are also known as quasi-primary fields.

III. ENERGY-MOMENTUM TENSOR

The energy momentum tensor $T_{\mu\nu}$ is an example of quasi-primary field that is not primary. Under a global conformal transformation, the classical energy momentum "field" transforms with a conformal dimension $h = \bar{h} = 2$. This tensor is of particularly import because of its relevance to conformal invariance. For a transformation $x^\mu \to x^\mu + \epsilon^\mu$, the variation in the action is:

$$\delta S = \frac{1}{d}\int \mathrm{d}^d x \; T^\mu_{\ \mu}\partial_\rho \epsilon^\rho \tag{8}$$

Therefore, the tracelessness of the energy momentum tensor implies conformal invariance. It can be shown classically that for translation, rotation, and scale invariant theories in $d > 2$, the trace of the energy momentum tensor vanishes. This is not the case for $d = 2$. While this may lead to one to believe that scale invariance implies conformal invariance for $d > 2$ and not $d = 2$ conformal field theorie, the exact opposite is the case. The subtlety lies in considering fluctuations (quantum/statistical), namely the vacuum expectation of T^μ_μ.

In $d = 2$, one can show by explicitly that the that the vacuum expectation value of the energy momentum tensor trace vanishes under translation, rotation, and scale invariance [2]. Said in a different way, if you have a fixed point that is translation and rotation invariant, the scale invariance assumed at fixed points implies conformal invariance. In fact, it is the vacuum tracelessness of $T_{\mu\nu}$ that will force us to regularize the energy momentum tensor and introduce the conformal anomaly.

We can see this readily by considering a free scalar field $h(r)$, with a classical action

$$S[h] = \frac{g}{4\pi}\int \mathrm{d}^2 r(\partial_\mu h)(\partial^\mu h) \tag{9}$$

is conformally invariant[3]. The energy-momentum zz component goes as $T_{zz} = -g(\partial_z h)^2$ so that

$$\langle h(z,\bar{z})h(0,0)\rangle \sim -\frac{1}{2g}\log\frac{z\bar{z}}{L^2} \tag{10}$$

which implies $\langle T_{zz}\rangle$ is (IR) divergent. This divergence can be regularized away by subtracting off the divergent piece. However, this regularization introduces a length scale and when one considers how this quantum energy momentum tensor changes under a local conformal transformation, we run into the c number.

$$T'(w) = \left(\frac{dw}{dz}\right)^{-2} T(z) + \frac{c}{12}\{z;w\} \tag{11}$$

where $\{z;w\}$ is the Schwarzian derivative [2]. This derivative vanishes for a global conformal transformation making it consistent with the fact that T is a quasi-primary field.

We see from the above example that the "central charge" c, is born from the regularization of the statistical/quantum fluctuations of the field. Note however that the introduction of a macroscopic length scale breaks the conformal invariance unless the coupling constants of the theory take a particular set of values–the RG fixed point[4]. It is this constraint that gives us so much information on two dimensional systems at criticality.

IV. CENTRAL CHARGE

The central charge can be calculated for a variety of theories with different kinds of fields. A free boson theory results in $c = 1$, a free fermion in $c = 1/2$ and a simple ghost in $c = -2$ [5]. In a n non-interacting boson theory, $c = n$. This implies that the central charge can be interpreted as the number of gapless degrees of freedom (Nambu-Goldstone modes) [5]. The gaplessness of these degrees of freedom follows since an introduction of a mass would break scale invariance. However, since c can be a non-integer, this intepretation should not be taken literally.

This c constant also shows up in other ways. If one considers defining a conformal field theory on a plane, the vacuum energy density is 0. However, if one considers a CFT on cylinder of circumference L, the vacuum energy density is

$$T_{zz} = -c\pi^2/6L^2 \tag{12}$$

The introduction of the periodicity condition with scale L changes the vacuum energy density proportional to the central charge c. This appearance of the central charge is known as the "Weyl anomaly". Just as before, the introduction of a macroscopic length scale in the form of the curvature breaks conformal invariance.

V. C-THEOREM

The c-theorem states that given a two dimensional model with rotational invariance, reflection positivity, and conservation of the stress tensor, there exists a function C of the coupling constants which is non-increasing along RG flow and is stationary at the fixed points. Reflection positivity is the condition that given an in state ψ by acting on the vacuum in negative Euclidean time,

$|\psi\rangle = \mathcal{O}(-t_{E_1})...\mathcal{O}(-t_{E_n})\,|0\rangle$, the norm is positive definite.

$$\langle\psi|\psi\rangle \geq 0 \tag{13}$$

If a Euclidean field theory is the result of a Wick rotated, unitary, Lorentzian theory, then it is reflection positive[6].

The c-theorem implies that $d = 2$ RG flow irreversibly and one may interpret the non-increasing property of the C function as the loss of information as one integrates out fast modes. However, we can go further:a simple analysis of the stationary values of this C function as precisely the central charge introduced in Section III. In conjunction with the discussion on c in Section IV, the c-theorem implies that when one goes from a UV fixed point to an IR fixed point, the gapless degrees of freedom greatly reduce.

Extending this theorem to higher dimensions have been met with difficulty. While the constraints are easy to satisfy locally, finding a suitable C function that is globally defined, finite at fixed points, and measurable solely in correlation functions is hard[5]. However, John Cardy's conjecture in 1988 of an analogous statement in four spacetime dimensions, the a-theorem, was proven through perturbation methods by Hugh Osborn [7] and non-perturbatively by Komargodski and Schimmer [8].

VI. CLASSIFICATION OF D=2 FIXED POINTS

To fully describe CFTs, and therefore fixed points, one requires the central charge c, the conformal dimension $h, \bar{h}$ and the fusion rules/operator product expansions of relevant operators. Various results over the past thirty years have resulted in a constraint on the possible values of the central charge in conformal field theories with reasonable assumptions. While we will not have the space or time to review over the derivation of the constraints nor present all such constraints, it is important to know that the most restrictive constraints follows from unitarity, modular invariance, and crossing symmetry. For minimal models, reflection positive/unitary theories with a finite number of primary fields, with $c < 1$ the central charge is constrained to be [9]:

$$c = 1 - \frac{6(m-m')^2}{mm'} \tag{14}$$

where m, m' are coprime integers. While one may think that these constraints are quite arbitrary, the first few allowed charges are quite familiar as $c = 1/2$ refers to the critical Ising Universality class (ϕ^4 model), $c = 7/10$ is the tricritical Ising universality class (ϕ^6 model). Note however $c = 4/5$ corresponds to two inequivalent universality classes: the tetracritical Ising model (ϕ^8 model) and the critical 3-state Potts model. While we have constrained the possible fixed points with the central charge, the critical exponents and thus universality classes depend non-trivially on the conformal dimensions of the fields in the theory. Regardless, it is still possible to enumerate all possible scaling dimensions given the central charge and thus we may classify every universality class in d = 2 systematically.

CONCLUSION

Conformal symmetry can be used in a variety of ways to understand symmetries. Much like other times of symmetry (translation invariance, time reversal symmetry, etc.), much can be gained by purely symmetric arguments. Conformal symmetry is quite strong in this regard as it can constrain or even determine the form of correlation functions, critical exponents, and fixed points. In this review, we traced the power of conformal symmetry $d = 2$ to highlight the importance of the central charge in characterizing critical field theories. However, in doing this beeline approach, we missed the Operator Product Expansion (OPE) formalism, the quantum symmetry algebra formulation of local conformal transformations, the insight gained from modular invariance via global conformal transformations, and more. Despite skipping these large topics, this review highlights how conformal transformations in two dimensions can heavily constrain a system, even those far from being actually conformally invariant via RG flow. While we have yet to see the light at the end of the tunnel in understanding CFT's of higher dimensions, the power of conformal invariance in two dimensions is undeniable.

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